

# Hilbert Space or Gel'fand Triplet. Time-Symmetric or Time-Asymmetric Quantum Mechanics

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Intrinsic microphysical irreversibility is the time asymmetry observed in exponentially decaying states. It is described by the semigroup generated by the Hamiltonian  $H$  of the quantum physical system, not by the semigroup generated by a Liouvillian  $L$  which describes the irreversibility due to the influence of an external reservoir or measurement apparatus. The semigroup time evolution generated by  $H$  is impossible in the Hilbert space (HS) theory, which allows only time-symmetric boundary conditions and a unitary group time evolution. This leads to problems with decay probabilities in the HS theory. To overcome these and other problems (nonexistence of Dirac kets) caused by the Lebesgue integrals of the HS, one extends the HS to a Gel'fand triplet, which contains not only Dirac kets, but also generalized eigenvectors of the self-adjoint  $H$  with complex eigenvalues ( $E_R - i\Gamma/2$ ) and a Breit-Wigner energy distribution. These Gamow states  $\psi^G$  have a time-asymmetric exponential evolution. One can derive the decay probability of the Gamow state into the decay products described by  $\Lambda$  from the basic formula of quantum mechanics  $\mathcal{P}(t) = \text{Tr}(|\psi^G\rangle\langle\psi^G|\Lambda)$ , which in HS quantum mechanics is identically zero. From this result one derives the decay rate  $\mathcal{P}(t)$  and all the standard relations between  $\mathcal{P}(0)$ ,  $\Gamma$ , and the lifetime  $\tau_R$  used in the phenomenology of resonance scattering and decay. In the Born approximation one obtains Dirac's Golden Rule.

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## 1. EXTRINSIC VS. INTRINSIC MICROPHYSICAL IRREVERSIBILITY

Irreversible time evolution of a microphysical system occurs extrinsically, as a result of interaction with an external system such as a reservoir or a measuring apparatus, or intrinsically as derived from the dynamics of the system. In the Hilbert space quantum mechanics, the time evolution described by the Hamiltonian must be time reversible, leading to a widespread

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conclusion that intrinsic irreversibility does not exist. Several authors, however, noticed examples of a microphysical arrow of time. Before discussing further these recent views, a brief exposition of extrinsic irreversibility is given.

In the case of a system  $\mathcal{S}$  interacting with a reservoir  $\mathcal{R}$ , the time evolution of the density operator  $\rho(t)$  is given by the master equation [Prigogine, 62; Davies, 76]

$$\frac{\partial \rho}{\partial t} = L\rho(t) \quad (1)$$

The Liouville operator  $L$  takes the form

$$L\rho(t) = -i[H, \rho(t)] + \delta H\rho(t) \quad (2)$$

Without the term  $\delta H\rho(t)$ , the equations (1) and (2) would be the von Neumann equations describing the reversible time evolution of a (closed) isolated quantum system, their solution being the unitary group evolution

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}; \quad -\infty < t < \infty \quad (3)$$

Since the Liouville operator of extrinsic irreversibility has the additional term  $\delta H\rho(t)$ , representing the effect of the reservoir  $\mathcal{R}$  on the system  $\mathcal{S}$ , the state no longer evolves according to the unitary group generated by the Hamiltonian as in (3). Instead, under certain additional conditions on the term  $\delta H\rho(t)$ , the integration of equation (2) leads to a semigroup evolution

$$\rho(t) = \Lambda(t)\rho(0), \quad \Lambda(t) = e^{Lt}, \quad \text{for } t \geq 0 \quad (4)$$

where  $\Lambda(t)$  is the Kossakowski–Lindblad semigroup [Ghirardi, 86; Antoniou, 93]. The semigroup time evolution describes extrinsic irreversibility because it applies to a combined system  $\mathcal{S} \otimes \mathcal{R}$ , where  $\mathcal{S}$  does not act on  $\mathcal{R}$ , but  $\mathcal{R}$  acts on  $\mathcal{S}$ .

The non-quantum mechanical term  $\delta H\rho$  in the right of equation (2) does not come from the intrinsic dynamics of  $\mathcal{S}$ . It is an empirical term, in the sense that every reservoir  $\mathcal{R}$  has its own way  $\delta H$  to act on  $\rho(t)$ . In this paper we shall not discuss extrinsic irreversibility, but intrinsic irreversibility.

In contrast to extrinsic irreversibility, intrinsic irreversibility is inherent to the dynamics of the quantum system; thus even closed (isolated) quantum systems can have irreversible time evolution. The unitary group evolution (3) is only a special case that applies to some (e.g., stationary), but not all quantum systems. The conventional opinion was that irreversible semigroup time evolution generated by the Hamiltonian of the quantum system is not possible. However, some suggestions of intrinsic irreversibility and time asymmetry in quantum physics have been mentioned in the past:

1. According to the work of R. Peierls and his school [Garcia-Calderon, 76; Hernandez, 84; Mondragon, 91; Peierls, 54; Peierls, 79], irreversibility is connected with the choice of initial and boundary conditions for the solutions of the Schrödinger equation. These new (purely outgoing) boundary conditions lead to microphysical irreversibility.

2. T. D. Lee [Lee, 81] explained that the time reverse of a decay process is highly improbable. The decay products have a fixed phase relationship. To reverse this decay process would require the preparation of a state consisting of two (or more) highly correlated incoming spherical waves with fixed relative phase. However, it is practically impossible to build an experimental apparatus that prepares two incoming waves with a fixed relative phase.

3. Ludwig [Ludwig, 83] noticed that a state  $\varphi$  must be prepared first (at  $t = 0$ ) before an observable  $|\psi(t)\rangle\langle\psi(t)|$  can be measured in it. This implies that a detector that is to register an observable in the state  $\varphi$  must be turned on during a time interval of positive time, i.e., at a time after the preparation apparatus (e.g., accelerator) has been turned on. This means the observable can only be translated to positive times, not to arbitrary negative times. Consequently, the time evolution operator of the observable should form only a semigroup  $U_+(t) = e^{iHt}$ ,  $t \geq 0$ . However, realizing that a time evolution semigroup generated by the Hamiltonian was not possible within the mathematics of the Hilbert space, Ludwig extrapolated this semigroup to all times  $-\infty < t < \infty$ .

4. Prigogine [Antoniou, 89; Petrosky, 91; Prigogine, 80; Prigogine, 92] had emphasized for a long time that irreversibility is intrinsic to the dynamics of the microsystem rather than caused by external influences of a reservoir or a measurement apparatus. Consequently, he demanded that irreversibility be connected with the Hamiltonian of the quantum system.

The most prominent example of intrinsic irreversibility is the time evolution of resonances. Resonances cannot be described within Hilbert space (HS) quantum mechanics as autonomous systems. We shall show in this paper that the same mathematics that had originally been introduced to justify the Dirac formalism and the nuclear spectral theorem, namely the rigged Hilbert space (RHS), also describes the irreversible decay of microsystems and allows for a mathematical theory of the quantum mechanical arrow of time.

## 2. HILBERT SPACE IDEALIZATION OF QUANTUM MECHANICS

The first attempt to put the ideas of quantum mechanics into some mathematical structure was achieved by Dirac [Dirac, 30]. Dirac introduced

the bras  $\langle E|$ ,  $\langle x|$ , the kets  $|E\rangle$ ,  $|x\rangle$ , and an algebra of observables generated by such operators as the Hamiltonian  $H$ , the position  $Q$ , the momentum  $P$ , etc., demanding that they fulfill the eigenvector equations

$$H|E\rangle = E|E\rangle \quad (5)$$

$$Q|x\rangle = x|x\rangle$$

In analogy to the basis vector expansion in the 3-dimensional space  $\bar{\mathbf{x}} = \sum_{i=1}^3 e_i x^i$ , Dirac postulated that the kets introduced form a complete basis system. This means that any vector  $\phi$  can be written as

$$\phi = \sum_{n=1}^{\infty} |E_n\rangle \langle E_n|\phi\rangle + \int_0^{\infty} dE |E\rangle \langle E|\phi\rangle \quad (6)$$

or as

$$\phi = \int_{-\infty}^{\infty} dx |x\rangle \langle x|\phi\rangle$$

Here  $|E_n\rangle$  are the eigenvectors of  $H$  with discrete eigenvalues  $E_n$  and  $|E\rangle$  are the eigenvectors of the Hamiltonian with eigenvalues  $E$  from a continuous set (for which we choose  $\mathbf{R}_+$ ). Comparing the basis vector expansion in the 3-dimensional case with Dirac's expansion, it is seen that the scalar products  $x^i = \mathbf{e}_i \cdot \mathbf{x}$  correspond to the factors  $\langle E_n|\phi\rangle$ ,  $\langle E|\phi\rangle$ , and  $\langle x|\phi\rangle$ . Consequently, these factors were understood as scalar products measuring the components of the vector  $\phi$  along the basis vectors formed by the eigenvectors of the observable. This interpretation is mathematically sound in the case of discrete eigenvectors  $|E_n\rangle$ , which are elements of some Hilbert space  $\mathcal{H}$ . However, for the continuous eigenvectors, the kets  $|E\rangle$  or  $|x\rangle$  are not in  $\mathcal{H}$  and the energy wavefunctions  $\langle E|\phi\rangle$  or the position wavefunctions  $\langle x|\phi\rangle$  are not scalar products, but generalizations thereof.

The mathematics available at that time could not encompass Dirac's formalism. In spite of this, it became the primary calculative tool for physicists, without having any rigorous mathematical foundation. In fact, it was not until Schwartz's theory of distributions [Schwartz, 50] that the Dirac delta function became mathematically defined, and Gel'fand's theory of RHS [Gel'fand, 64; Maurin, 68] that Dirac's kets  $|E\rangle$  and  $|x\rangle$  received a mathematical interpretation.

After Dirac's ideas, the first attempt at a rigorous mathematical theory for quantum mechanics was provided by Weyl [Weyl, 28] and von Neumann [Neumann, 31] using the mathematics that was available at that time: the mathematics of the Hilbert space.

A Hilbert space (HS) is the completion of a linear space with scalar product  $\langle \phi|F\rangle = (\phi|F)$ , which defines the norm  $\|\phi\| = \sqrt{(\phi, \phi)}$ . A linear

space  $\Phi$  with scalar product is incomplete if not all Cauchy sequences have limit elements in that space. Physicists usually do not worry about the completion of their Hilbert space; mathematicians call such spaces pre-Hilbert spaces. The Hilbert space  $\mathcal{H}$  is obtained by completing  $\Phi$ , i.e., appending to  $\Phi$  all (limit elements of) Cauchy sequences. According to the HS formulation of quantum mechanics [Neumann, 31] there is a one-to-one correspondence between vectors in the Hilbert space and pure physical states, and between self-adjoint operators on the Hilbert space and observables.

The wave function  $\langle x|\psi\rangle = \psi(x)$  gives the probability  $|\psi(x)|^2\Delta x$  to detect the particle in the position interval  $\Delta x$ . The wave function  $\phi(E) = \langle E|\phi\rangle$ , in the energy representation, describes the energy distribution of, e.g., a particle beam produced by the accelerator [ $|\phi(E)|^2$  represents the energy resolution of the experimental apparatus]. Physicists always associate smooth functions with these quantities. In the Hilbert space formulation of quantum mechanics, these wavefunctions are elements of the space of Lebesgue square-integrable functions on the real line  $L^2(\mathbf{R})$  [Maurin, 80]. Elements of  $L^2(\mathbf{R})$  are classes of Lebesgue square-integrable functions  $\{\psi(x)\}$  or  $\{\phi(E)\}$  that may vary widely on a set of Lebesgue measure zero (e.g., all rational numbers). One and the same wavefunction  $\phi(E)$  can be given by any function in the class  $\{\phi(E)\}$ , not only by the smooth function of this class. In addition there are classes that do not contain a smooth function while still being Lebesgue square-integrable. This feature contradicts physical intuition since the wavefunction connected with the experimental apparatus is always thought of as a smooth function  $|\phi(E)|^2$ . While it is true that the space of smooth (infinitely differentiable and rapidly decreasing) functions  $\mathcal{S}$  is a dense subset of  $L^2(\mathbf{R})$ ,  $\mathcal{S}$  is not complete in the norm defined by the scalar product. Insisting on a complete topological vector space, mathematicians chose the Hilbert space  $L^2$  for quantum mechanics because the space  $\mathcal{S}$ , whose completion is defined not by one norm, but by a countable number of norms  $\|\cdot\|_1, \|\cdot\|_2, \dots, \|\cdot\|_p, \dots$ , did not exist at that time.

To each smooth function  $\psi^{\text{smooth}}(x) \in \mathcal{S}$  one can always find a class of Lebesgue square-integrable functions  $\{\psi(x)\}$  to which  $\psi^{\text{smooth}}(x)$  belongs, i.e.,  $\mathcal{S} \subset L^2$ , but not vice versa: there are classes of Lebesgue square-integrable functions  $\{\chi(x)\} \in L^2$  which do not contain any smooth function, because the space of smooth functions  $\mathcal{S}$  is not complete with respect to the norm  $\|\psi\|^2 = \int |\psi(x)|^2 dx$ . If experiments provide only smooth wavefunctions (they measure only in a finite set of intervals and interpolate smoothly between these intervals), then the space  $\mathcal{S}$  should be sufficient for all states  $\psi$  connected with experimental apparatuses. Thus the complete Hilbert space  $L^2$  is too big.

On the other hand, the HS does not contain Dirac's kets and bras, because the eigenstates of the continuous spectrum are not in the HS and certainly the HS does not possess a complete basis system in the sense of (6). However

these kets, e.g., the scattering “states”  $|\bar{p}\rangle$  with momentum eigenvalue  $\bar{p}$ , have been very useful for scattering experiments. Thus the Hilbert space  $L^2$  is also too small, since it does not contain these scattering states. We shall therefore seek a formulation which will overcome these problems. As an unexpected bonus, this new formulation will also contain vectors that represent exponentially decaying states and will describe irreversible processes like quantum decays.

### 3. PROBLEMS WITH QUANTUM DECAY

Most practical computations in physics do not rely on the completeness property of the Hilbert space and use only the pre-Hilbert space. However, when investigating general properties of decay, the full mathematical structure of the Hilbert space has led to general results which are not desirable for a theory of decaying phenomena.

The first property of the HS formulation to note is that symmetry groups, like the Galileo group and the Poincaré group, are represented by unitary operators. In the Hilbert space, for a system described by a Hamiltonian  $H$ , the time evolution is given by a unitary group:

$$\phi(t) = e^{-iHt}\phi(0) = U^\dagger(t)\phi(0), \quad -\infty < t < \infty \quad (7)$$

and is reversible. Therefore, any physical state in the HS can be evolved to any instant in the past and in the future. This defies physical intuition regarding the evolution of resonance states backward to instances before their production. Microphysical irreversibility, as exemplified by the time evolution of resonances or decaying states, is ruled out by this unitary group evolution.

Another important feature of HS quantum mechanics relevant to decay processes is that no exponential decay law can be obtained within the framework of Lebesgue square-integrable functions [Khalfin, 72]. This result deals with the time behavior of the survival probability of a given state  $\phi(0)$ . The survival probability is the probability to find, at any given time  $t$ , the state  $\phi(0)$  in the time-evolved state  $\phi(t) = e^{-iHt}\phi(0)$ :

$$\mathcal{P}_S(t) = |\langle \phi(0) | e^{-iHt} \phi(0) \rangle|^2 \quad (8)$$

Khalfin’s theorem states that there is no HS vector  $\phi(0)$  for which (8) obeys the exponential law. This “deviation from the exponential law” has at least so far not been confirmed experimentally. But since the mathematics of the Hilbert space cannot predict the magnitude of such a deviation, it will always remain an untestable mathematical prediction (because the deviation could always be smaller than the available experimental accuracy). Therefore the more practical attitude is to find a mathematical formulation that upholds the empirical exponential law for the resonance state and attribute whatever

deviations may be observed in the future to the admixture of some background. The description of exponential decay can be best accomplished, as envisioned by Gamow [Gamow, 28], if one uses an eigenvector  $\psi^G \equiv |E_R - i\Gamma/2\rangle$  of the self-adjoint Hamiltonian that has a complex eigenvalue  $E_R - i\Gamma/2$ ,

$$H|E_R - i\Gamma/2\rangle = (E_R - i\Gamma/2)|E_R - i\Gamma/2\rangle \quad (9)$$

As mentioned above, such vectors do not exist in the HS, but they exist in the RHS.

The strongest evidence for the inadequacy of the HS in the description of decay phenomena is that the decay probability is zero for all time if it is zero on a finite time interval. The decay probability is the probability for the transition from a state  $\phi(t) = e^{-iHt}\phi(0) = U^t(t)\phi(0)$  into the decay products described by the subspace  $\Lambda\mathcal{H} \subset \mathcal{H}$ , where  $\Lambda$  is the projection operator on the subspace of noninteracting decay products. This decay probability is given by

$$\mathcal{P}_\Lambda(t) = \text{Tr}(\Lambda|\phi(t)\rangle\langle\phi(t)|) \quad (10)$$

Since the decay of a prepared quasistationary state starts at a finite time  $t > t_2 > -\infty$ , the probability to detect the observable  $\Lambda$  in the state  $|\phi(t)\rangle$  during a time interval  $-\infty \leq t_1 < t_2$  should be zero. In other words,

$$\int_{t_1}^{t_2} \mathcal{P}_\Lambda(t) dt = \int_{t_1}^{t_2} \langle\phi(t)|\Lambda|\phi(t)\rangle dt = 0 \quad (11)$$

Then it follows from Hegerfeldt's theorem [Hegerfeldt, 94] that, if  $\phi(t) = e^{-iHt}\phi(0) \in \mathcal{H}$ , with the Hamiltonian  $H$  being self-adjoint and semibounded, then  $\mathcal{P}_\Lambda(t) = 0$  for (almost) all  $t$  (future  $t > t_2$  and past  $t < t_1$ ). This means that, according to the Hilbert space formulation, there can be no decay of a state  $\phi(t)$  which has been produced at any finite time  $t_2 (\neq -\infty)$ .

Summarizing, in HS quantum mechanics the time evolution is reversible, there is no exponential decay law, and the decay probability is identically zero. These are the underlying reasons that in practical calculations resonances could not be described in the Hilbert space. Instead, resonances were successfully described by "effective theories" as eigenvectors of some finite-dimensional complex Hamiltonians [Lee *et al.*, 57]. As will be seen, the RHS provides a mathematical theory which can overcome the problems of the HS theory. In addition, the RHS will contain a finite-dimensional subspace in which the effective theories reappear as truncations of a complex basis vector expansion.

#### 4. RIGGING THE HILBERT SPACE INTO A GEL'FAND TRIPLET

A rigged Hilbert space is constructed on the structure of a linear space  $\Psi$  with a scalar product  $\langle \phi | F \rangle = (\psi, |F\rangle)$  through the completion of the space  $\Psi$  with respect to different topologies. The completion of  $\Psi$  with respect to a topology  $\tau$  contains all the limit points of the Cauchy sequences in the respective topology. The Hilbert space  $\mathcal{H}$  is the completion of  $\Psi$  with respect to the norm topology  $\tau_{\mathcal{H}}$ . The space  $\Phi$  is defined as the completion of  $\Psi$  with respect to a topology  $\tau_{\Phi}$ , stronger than the norm topology. Since  $\Phi$  and  $\mathcal{H}$  are the completions of the same space  $\Psi$ ,  $\Phi$  will be dense in  $\mathcal{H}$  with respect to the topology of the Hilbert space. The topology  $\tau_{\Phi}$  of the space  $\Phi$  is given by an infinite number of norms chosen such that the algebra of observables in the space  $\Phi$  becomes an algebra of continuous operators.  $\Phi^{\times}$  (and  $\mathcal{H}^{\times}$ ) denote the space of continuous antilinear functionals over the space  $\Phi$  (and  $\mathcal{H}$ ). These antilinear functionals  $F(\phi)$  are denoted as  $F(\phi) = \langle \phi | F \rangle$  [or  $F(h) = \langle h | F \rangle$ ] and are defined over the set  $\phi \in \Phi$  (or  $h \in \mathcal{H}$ ). There are more functionals  $|F\rangle$  in  $\Phi^{\times}$  than  $|F\rangle$  in  $\mathcal{H}^{\times}$ , and since  $\mathcal{H}^{\times} = \mathcal{H}$  (Frechet–Riesz theorem), one has constructed a triplet of spaces, called a *Gel'fand triplet* or RHS [Gel'fand, 64; Maurin, 68]:

$$\Phi \subset \mathcal{H} = \mathcal{H}^{\times} \subset \Phi^{\times} \quad (12)$$

The space  $\Phi^{\times}$  is an extension of the Hilbert space  $\mathcal{H}$ . The topology  $\tau_{\Phi}$  can be chosen such that  $\Phi^{\times}$ , unlike the Hilbert space  $\mathcal{H}$ , contains “eigenvectors” of a self-adjoint operator with eigenvalues belonging to the continuous spectrum, e.g., the Dirac kets. In addition  $\Phi^{\times}$  contains “eigenvectors” of self-adjoint operators with complex eigenvalues. The Dirac bra–ket  $\langle \phi | F \rangle$  is just an extension of the scalar product  $(\phi, F)$  in  $\mathcal{H}$  [Bohm, 79].

In a scattering experiment one defines two sets of rigged Hilbert spaces, one for the prepared in-states  $\Phi_{-}$ , and the other for the observed (detected) out-states  $\Phi_{+}$  [Bohm, 89]:

$$\Phi_{-} \subset \mathcal{H} = \mathcal{H}^{\times} \subset \Phi_{-}^{\times} \quad (13)$$

and

$$\Phi_{+} \subset \mathcal{H} = \mathcal{H}^{\times} \subset \Phi_{+}^{\times}$$

where  $\Phi = \Phi_{-} + \Phi_{+}$  and  $\Phi_{-} \cap \Phi_{+} \neq 0$ . A vector  $\phi^{+} \in \Phi_{-}$  is what is prepared as  $\phi^{in}$  outside the interaction region, while a vector  $\psi^{-} \in \Phi_{+}$  is what is registered as  $\psi^{out}$  outside the interaction region.

The topology in the space  $\Phi$ , and equivalently in the spaces  $\Phi_{+}$  and  $\Phi_{-}$ , is always chosen in such a way that the operators representing the observables are continuous (so bounded) operators on  $\Phi$  (with respect to the



topology  $\tau_{\Phi}$ ). In the Hilbert space the observables cannot be represented by continuous operators (with respect to  $\tau_{\mathcal{H}}$ ). For example, in the Hilbert space, if the operators  $P$  and  $Q$  satisfy the Heisenberg commutation relation, then they cannot be both continuous operators in  $\mathcal{H}$ , and in the standard representation neither  $P$  nor  $Q$  is a continuous operator. As  $\Phi$  is a dense subspace of the Hilbert space, all the operators that are used in the Hilbert space can be redefined in  $\Phi$  as restrictions to the space  $\Phi$ . For each  $\tau_{\Phi}$ -continuous operator  $A$  on  $\Phi$  one can define its conjugate operator  $A^{\times}$ , as an extension of the HS adjoint operator  $A^{\dagger}$ .

$$\langle A\phi|F\rangle = \langle \phi|A^{\times}|F\rangle \quad \text{for all } \phi \in \Phi, \quad F \in \Phi^{\times} \quad (14)$$

As a result, we obtain a triplet

$$A^{\dagger}|_{\Phi} \subset A^{\dagger} \subset A^{\times} \quad (15)$$

It should be noted that the conjugate operator  $A^{\times}$  can only be defined for a  $\tau_{\Phi}$ -continuous operator  $A$ , and, consequently, is a continuous operator on  $\Phi^{\times}$ . The generalized eigenvector  $|F\rangle$  of a continuous operator  $A$  is defined by the following relation:

$$\langle A\phi|F\rangle = \langle \phi|A^{\times}|F\rangle = \omega\langle \phi|F\rangle \quad \text{for all } \phi \in \Phi \quad (16)$$

Ignoring the arbitrary vectors  $\phi$ , this is often also written as

$$A^{\times}|F\rangle = \omega|F\rangle \quad (17)$$

or as

$$A|F\rangle = \omega|F\rangle$$

if  $A$  is essentially self adjoint. This method makes it possible to describe "eigenstates" that cannot exist in the Hilbert space. Some of the generalized eigenvectors are going to be the ordinary eigenvectors of the essentially self-adjoint operator in the Hilbert space. But not all generalized eigenvectors are elements of the Hilbert space. In particular, the Dirac kets, which describe the scattering states, are generalized eigenvectors with eigenvalues belonging to the continuous spectrum and are not in  $\mathcal{H}$ . The Gamow vectors, which describe the states with an irreversible time evolution, are also generalized eigenvectors which are not in  $\mathcal{H}$ , but their complex eigenvalues do not belong to the Hilbert space spectrum of the Hamiltonian. The choice of  $\Phi$ , given by the choice of the topology  $\tau_{\Phi}$ , determines which set of generalized eigenvectors is possible for a given operator  $A$  in  $\mathcal{H}$ . This choice of the spaces  $\Phi$ ,  $\Phi_+$ , and  $\Phi_-$  is made using physical arguments related to causality and initial and boundary conditions. The initial conditions are determined by the setup of the experiment. In the RHS formulation of quantum mechanics one uses

the same dynamical equations as in the Hilbert space formalism, while the initial (boundary) conditions are different from the HS boundary conditions. The space  $\mathcal{H}$  in the HS formalism describes all physical systems; for each particular system only a dense subspace is used for practical calculations. The spaces  $\Phi$ ,  $\Phi_+$ , and  $\Phi_-$  are specific for the particular physical system under consideration.

## 5. GAMOW VECTORS AND THEIR PROPERTIES

The rigged Hilbert space was developed in order to accommodate Dirac's kets and bras into a consistent mathematical structure, but the structure created for Dirac's formalism provided a mathematical description for the states with an irreversible time evolution, too. In the Hilbert space, an irreversible process is possible only for an open system under the influence of an external reservoir. There are no vectors in  $\mathcal{H}$  which can represent isolated microphysical states that can evolve irreversibly in time. In the RHS, decaying states which are described by the Gamow vectors  $|z_R^- \rangle \equiv |E_R - i\Gamma/2^- \rangle$  evolve irreversibly in time by a semigroup generated by the Hamiltonian.

The following are the properties of Gamow vectors describing decaying states:

1. They are generalized eigenvectors of the Hamiltonian associated with the complex eigenvalue  $E_R - i\Gamma/2$  (where  $E_R$  and  $\Gamma$  are interpreted as the resonance energy and the width of the resonance, respectively); i.e., the following equation holds:

$$H^\times |z_R^- \rangle = (E_R - i\Gamma/2) |z_R^- \rangle \quad (18)$$

as a functional equation over all  $\psi^- \in \Phi_+$  [in the sense of (17)].

2. They are derived as functionals from the resonance pole term at  $z_R = E_R - i\Gamma/2$  in the second sheet of the analytically continued  $S$ -matrix.

3. They have a Breit–Wigner energy distribution

$$\langle^- E | \psi^G \rangle = i\sqrt{\Gamma/2\pi} \frac{1}{E - (E_R - i\Gamma/2)}, \quad -\infty_{\text{II}} < E < +\infty \quad (19)$$

(where the negative values of  $E$  are in the second Riemann sheet of the  $S$ -matrix).

4. They are members of a basis system (like the Dirac kets  $|E\rangle$ ), i.e., every prepared state vector  $\phi^+ \in \Phi_-$  can be expanded as [Bohm, 97]

$$\begin{aligned} \phi^+ &= \sum_{n=1}^{\infty} |E_n\rangle (E_n | \phi^+) + \sum_{i=1}^N |\psi_i^G\rangle \langle \psi_i^G | \phi^+ \rangle \\ &+ \int_0^{-\infty_{\text{II}}} dE |E^+\rangle \langle^+ E | \phi^+ \rangle \end{aligned} \quad (20)$$

(where  $-\infty_{II}$  indicates that the integration along the negative real axis is in the second Riemann sheet). In contrast, the Dirac basis system expansion (Nuclear Spectral Theorem of the RHS) is given by

$$\phi^+ = \sum_{n=1}^{\infty} |E_n\rangle \langle E_n| \phi^+ + \int_0^{+\infty} dE |E^+\rangle \langle E| \phi^+ \quad (21)$$

where

- $|E_n\rangle, n = 1, 2, \dots, \infty$  are the stable eigenstates (bound state poles)
- $|\psi_i^G\rangle = |E_i - i\Gamma_i/2\rangle \sqrt{2\pi\Gamma_i}$  are the  $N$  decaying (Gamow) states (resonance poles)
- $|E^+\rangle, 0 \leq E < \infty$  (Hilbert space spectrum) are the Dirac scattering states
- $|E^+\rangle, -\infty_{II} < E \leq 0$ , are the latter's analytic continuation to the negative real axis on the second sheet.

The important feature of the so-called complex spectral resolution (20) is that the resonance states  $\psi_i^G$  appear on the same footing as the bound states  $|E_n\rangle$ , but together with the bound states they do not form a complete system; there is in addition a "background term."

5. The time evolution, in general, is given by a semigroup generated by the Hamiltonian for  $t \geq 0$  (a corresponding semigroup with  $t \leq 0$  applies to the exponentially growing Gamow vector  $\bar{\psi}^G \in \Phi_+^\times$ , which is associated with the  $S$ -matrix pole at  $z_R = E_R + i\Gamma/2$ ) [Bohm, 79]. The unitary time evolution group applies only to the common Hilbert subspace of  $\Phi_+^\times$  and  $\Phi_-^\times$ . The time evolution of the decaying Gamow state in particular is given by an exponential law:

$$\begin{aligned} e^{-iH^\times t} |\psi^G\rangle \langle \psi^G| e^{iHt} \\ = e^{-i(E_R - i\Gamma/2)t} |\psi^G\rangle \langle \psi^G| e^{i(E_R + i\Gamma/2)t} = e^{-\Gamma t} |\psi^G\rangle \langle \psi^G| \end{aligned} \quad (22)$$

for  $t \geq 0$  only. This is understood as a functional equation over the space of  $\psi^- \in \Phi_+$ . This time evolution is irreversible because  $e^{-iH^\times t} (t \geq 0)$  is a semigroup.

## 6. DECAY PROBABILITY AND DECAY RATE IN RHS

In the RHS, the description of irreversible processes becomes possible. Decaying states are described by Gamow vectors. A process in which a microphysical state evolves in time and decomposes into a set of decay products will be described as the transition of a Gamow vector  $\psi^G$  into a set of interaction free decay products.

In a decay experiment, the decaying state and the set of detected decay products are described by different Hamiltonians. While the detected states

evolve in time according to the free Hamiltonian  $H_0$  (since they are supposed to be detected far away from the interaction zone), the decaying state is a generalized eigenvector of the exact Hamiltonian  $H = H_0 + V$ , where  $V$  is the interaction responsible for the decay. The eigenkets of the free Hamiltonian  $|E, b\rangle$  are assumed to be mapped into the eigenkets of the exact Hamiltonian  $|E, b^-\rangle$  by the Lippmann–Schwinger equation

$$|E, b^-\rangle = |E, b\rangle + \frac{1}{E - H - i\epsilon} V|E, b\rangle \quad (23)$$

Examples of decay processes are the radiative decay of an excited atom into its ground state ( $A^* \rightarrow A + \gamma$ ) with the emission of a photon, and the decay of a  $K$ -meson ( $K^0 \rightarrow \pi^+\pi^-$ ) into two pions.

The decay rate  $\mathcal{P}(t)$  of the  $\psi^G(t)$  into the final noninteracting decay products can be calculated as a function of time and leads to an exact Golden Rule [with the natural linewidth given by a Breit–Wigner energy distribution (19)]. In the Born approximation the Gamow vector  $\psi^G$  goes into  $f^d$  ( $\psi^G \rightarrow f^d$ , which is an eigenvector of  $H_0 = H - V$ ;  $E_R \rightarrow E_d$  and  $\Gamma/E_R \rightarrow 0$ ) and the decay rate goes into Fermi’s Golden Rule.

The time evolution of the “pure Gamow state” with resonance parameters  $(E_R, \Gamma)$ , initially described by the statistical operator  $W(0) = |\psi^G\rangle\langle\psi^G|$  is given, according to (22), by

$$W(t) = e^{-iHt}W(0)e^{iHt} = e^{-\Gamma t}W(0) \quad \text{for } t \geq 0 \quad (24)$$

This is mathematically defined only as a functional over  $\psi^- \in \Phi_+$ , where  $\Phi_+$  is defined as the space connected with the decay products (“out-states”). This means that only  $\langle\psi^-|W(t)|\psi^-\rangle$  makes sense mathematically.

The interaction-free decay products are described by the projection operator  $\Lambda$  onto the space of the physical states of all the noninteracting decay products:

$$\Lambda = \int dE \sum_b |E, b\rangle\langle E, b| \quad (25)$$

where  $|E, b\rangle$  are the eigenvectors of the free Hamiltonian

$$H_0|E, b\rangle = E|E, b\rangle \quad (26)$$

and  $b$  stands for all the possible labels of these eigenvectors. If the system is described by a complete set of commuting operators  $B_1, B_2, \dots, B_N$ , then  $b$  will be given by the set  $b_1, b_2, \dots, b_N$  of quantum numbers labeling the degeneracy of the energy  $E$  [the  $b$ ’s can be the quantum numbers for the orbital angular momentum, the photon polarization  $\lambda_\gamma$ , some other intrinsic quantum numbers like charges or channel labels, or the momentum directions

( $\theta_k, \phi_k$ ) of the decay products]. We will use the index  $b$  for the whole set of quantum numbers in order to simplify the formulas since the choice of these labels will not change the results. For example, in (25), for the process  $A^* \rightarrow A + \gamma$ ,

$$|E, b\rangle = |E, \theta_k, \phi_k, \lambda_\gamma, \dots\rangle$$

and

$$\sum_b = \sum_{\lambda_\gamma} \int d \cos \theta_k d\phi_k \sum_{\dots}$$

where the dots stand for the quantum numbers of the atomic states.

The decay probability is the expectation value of the operator  $\Lambda$  for the interaction-free decay products in the state  $W(t)$  of the decaying state. Therefore it is given by the general formula for the expectation value of an observable  $\Lambda$  in a state  $W(t)$ :

$$\mathcal{P}(t) = \text{Tr}(\Lambda W(t)) \quad (27)$$

As explained in section 3, in the Hilbert space one can prove [with the only assumption that  $H$  is self-adjoint and semibounded, which must always be the case (stability of matter)] that  $\mathcal{P}(t)$  is either identically zero for all times or it has been already different from zero in a time interval starting at  $t = -\infty$ . This means that in the HS one predicts no decay for any state that has been prepared at a particular time  $t_0$  ( $\neq \infty$ ). In the RHS one can derive from (27), with  $W(0)$  given by  $|\psi^G\rangle\langle\psi^G|$  and  $\Lambda$  by (25), that

$$\mathcal{P}(t) = 1 - e^{-\Gamma t} \int dE \sum_b |\langle E, b|V|\psi^G\rangle|^2 \frac{1}{(E - E_R)^2 + (\Gamma/2)^2}$$

for  $t \geq 0$  (28)

In this derivation one uses (18), (24), and the Lippmann–Schwinger equation (23) and one chooses as boundary conditions  $\mathcal{P}(t = \infty) = 1$  (meaning that after a long enough time all the decay products have decayed and their decay products have been measured) and  $\mathcal{P}(t = 0) = 0$  (so that no decay product is measured before the preparation of the decaying state is completed at time  $t = t_0 = 0$ ).

An exact Golden Rule for the decay rate is obtained by taking the time derivative of the transition probability  $\mathcal{P}$  given in (28):

$$\dot{\mathcal{P}}(t) = 2 \pi e^{-\Gamma t} \int dE \sum_b |\langle E, b|V|\psi^G\rangle|^2 \frac{\Gamma/2\pi}{(E - E_R)^2 + (\Gamma/2)^2} \quad (29)$$

The decay rate  $\mathcal{P}(t)$  has a Breit–Wigner distribution, whose width is  $\Gamma$ . This is an exact formula from which one can obtain in the Born approximation Fermi’s Golden Rule if one inserts for the state  $\psi^G$  the noninteracting state  $f^d$  with  $H_0 f^d = E_d f^d$ . The Born approximation is defined by

$$\begin{aligned} \langle E, b|V|\psi^G\rangle &\rightarrow \langle E, b|V|f^d\rangle \\ E_R &\rightarrow E_d \\ \Gamma/2E_R &\rightarrow 0 \\ \frac{\Gamma/2\pi}{(E - E_R)^2 + (\Gamma/2)^2} &\rightarrow \delta(E - E_R) \end{aligned} \quad (30)$$

In this approximation (30), the initial decay rate is obtained from (29) as

$$\mathcal{P}(0) = 2\pi \int dE \Sigma |\langle E, b|V|f^d\rangle|^2 \delta(E - E_R) \quad (31)$$

This is the standard Golden Rule for the transition from an excited noninteracting state  $f^d$  into the set of all noninteracting decay products.

On the other hand, using the condition  $\mathcal{P}(0) = 0$ , one obtains from (28) in the limit (30)

$$\Gamma = 2\pi \int dE \Sigma |\langle E, b|V|f^d\rangle|^2 \delta(E - E_R) \quad (32)$$

Comparing this with (31), one obtains that

$$\mathcal{P}(0) = \Gamma \quad (33)$$

From the exponential decay law in (28), for the survival probability  $1 - \mathcal{P}(t) = e^{-\Gamma t}$ , or from the exponential law in (29) for the decay rate, one obtains that

$$\Gamma = \frac{1}{\tau_R} \quad (34)$$

where  $\tau_R$  is the lifetime of the resonance state. The results (31), (33), and (34) and the identification of  $\Gamma$  with the imaginary part of the resonance pole position  $z_R$  of the analytically continued  $S$ -matrix are the standard relations used in the analysis of resonance scattering and decay phenomena. They have been justified by various more or less heuristic arguments, in particular also by making use of the exponential law for the survival probability. But they have not been derived from the basic formula (27) for the probabilities in quantum mechanics, and they could not have been derived from this formula because, applied to the probabilities of decay, this formula is identically zero

in the HS formulation [Hegerfeldt, 94], as mentioned in Section 3. The quantity that had been missing from the HS formulation, and which is needed to provide the theoretical link between these important empirical formulas and the basic formula for probabilities (27), is the Gamow vector. Gamow vectors allow the description of resonances as elementary particles in very much the same way as stable particles are described, either as poles of the  $S$ -matrix or as energy eigenstates, except that for the Gamow states the energy is the complex number  $E_R - i\Gamma/2$  and this requires the mathematics of the RHS.

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